

Contents

1	Surface Preliminaries	1
1.1	Surfaces	1
1.2	Euclidean Space	2
1.3	Open Sets	3
1.4	Functions and Their Properties	9
1.5	Continuity	11
1.6	Problems	16
2	Surfaces	19
2.1	The Definition of a Surface	19
2.2	Examples of Surfaces	19
2.3	Spheres as Surfaces	22
2.4	Surfaces with Boundary	23
2.5	Closed, Bounded, and Compact Surfaces	24
2.6	Equivalence Relations and Topological Equivalence	24
2.7	Homeomorphic Spaces	26
2.8	Invariants	27
2.9	Problems	28
3	The Euler Characteristic and Identification Spaces	31
3.1	Triangulations and the Euler Characteristic	31
3.2	Invariance of the Euler Characteristic	35
3.3	Identification Spaces	37
3.4	ID Spaces as Surfaces	39
3.5	Abstract Topological Spaces	40
3.6	The Quotient Topology	42
3.7	Further Examples of ID Spaces	43
3.8	Triangulations of ID Spaces	45
3.9	The Connected Sum	46

3.10	The Euler Characteristic of a Compact Surface with Boundary	47
3.11	Problems	48
4	Classification Theorem of Compact Surfaces	51
4.1	The Geometry of the Projective Plane and the Klein Bottle	51
4.2	Orientable and Nonorientable Surfaces	54
4.3	The Classification Theorem for Compact Surfaces	56
4.4	Compact Surfaces Have Finite Triangulations	57
4.5	Proof of the Classification Theorem	58
4.6	Problems	61
5	Introduction to Group Theory	63
5.1	Why Use Groups?	63
5.2	A Motivating Example	64
5.3	Definition of a Group	64
5.4	Examples of Groups	65
5.5	Free Groups, Generators, and Relations	70
5.6	Free Products	73
5.7	Problems	74
6	Structure of Groups	77
6.1	Subgroups	77
6.2	Direct Products of Groups	78
6.3	Homomorphisms	80
6.4	Isomorphisms	83
6.5	Existence of Homomorphisms	84
6.6	Finitely Generated Abelian Groups	87
6.7	Problems	89
7	Cosets, Normal Subgroups, and Quotient Groups	91
7.1	Cosets	91
7.2	Lagrange's Theorem and Its Consequences	94
7.3	Coset Spaces and Quotient Groups	95
7.4	Properties and Examples of Normal Subgroups	96
7.5	Coset Representatives	98
7.6	A Quotient of a Dihedral Group	98
7.7	Building up Finite Groups	99
7.8	An Isomorphism Theorem	101
7.9	Problems	101
8	The Fundamental Group	105
8.1	Paths and Loops on a Surface	105
8.2	Equivalence of Paths and Loops	106
8.3	Equivalence Classes of Paths and Loops	107

8.4	Multiplication of Path and Loop Classes	108
8.5	Definition of the Fundamental Group	110
8.6	Problems	113
9	Computing the Fundamental Group	115
9.1	Homotopies of Maps and Spaces	115
9.2	Computing the Fundamental Group of a Circle	123
9.3	Problems	125
10	Tools for Fundamental Groups	127
10.1	More Fundamental Groups	127
10.2	The Degree of a Loop	129
10.3	Fundamental Group of a Circle—Redux	132
10.4	The Induced Homomorphism on Fundamental Groups	134
10.5	Retracts	137
10.6	Problems	139
11	Applications of Fundamental Groups	141
11.1	The Fundamental Theorem of Algebra	141
11.2	Further Applications of the Fundamental Group	145
11.3	Problems	149
12	The Seifert–Van Kampen Theorem	151
12.1	The Fundamental Group of a Wedge of Circles	151
12.2	The Seifert–Van Kampen Theorem: First Version	153
12.3	More Fundamental Groups	155
12.4	The Seifert–Van Kampen Theorem: Second Version	156
12.5	The Fundamental Group of a Compact Surface	157
12.6	Even More Fundamental Groups	159
12.7	Proof of the Second Version of the Seifert–Van Kampen Theorem	160
12.8	General Seifert–Van Kampen Theorem	161
12.9	Groups as Fundamental Groups	161
12.10	Problems	163
13	Introduction to Homology	165
13.1	The Idea of Homology	165
13.2	Chains	166
13.3	The Boundary Map	168
13.4	Homology	169
13.5	The Zeroth Homology Group	171
13.6	Homology of the Klein Bottle	172
13.7	Homology and Euler Characteristic	173
13.8	Homology and Orientability	174
13.9	Smith Normal Form	175

13.10	The Induced Map on Homology	178
13.11	Problems	180
14	The Mayer–Vietoris Sequence	181
14.1	Exact Sequences	181
14.2	The Mayer–Vietoris Sequence	183
14.3	Homology of Orientable Surfaces	186
14.4	The Jordan Curve Theorem	188
14.5	The Hurewicz Map	189
14.6	Problems	191
	Correction to: The Seifert–Van Kampen Theorem	C1
	Appendix A: Topological Notions	193
	Appendix B: A Brief Look at Singular Homology	197
	Appendix C: Hints for Selected Problems	201
	References	203
	Index	207