Contents

1 Introduction **1**

1.1 Some Simple ODEs 1

1.1.1 Examples 1

1.1.2 Descriptive Concepts 2

1.2 Solutions of ODEs 4

1.2.1 Examples and Discussion 4

1.2.2 Geometric Interpretation of Solutions 5

1.3 Snippets of Solution Techniques 6

1.3.1 Computer Solutions 6

1.3.2 Linear Equations with Constant Coefficients 7

1.3.3 First-Order Linear Equations 8

1.3.4 Separable Equations 8

1.4 Physically Based Second-Order ODEs 10

1.4.1 Linear Spring–Mass Systems 10

1.4.2 Nonlinearity, Part I: The Restoring Force 12

1.4.3 Energy 14

1.4.4 Nonlinearity, Part II: The Frictional Force 17

1.5 Systems of ODEs . . . 17
1.6 Topics Covered in This Book . . . 20
1.6.1 General Overview . . . 20
1.6.2 A Case Study in the Qualitative Theory of ODEs . . . 21
1.7 Exercises . . . 24
1.7.1 Core Exercises . . . 26
1.7.2 Computational Exercises . . . 29
1.7.3 Anticipatory Exercises . . . 31
1.7.4 PHD Exercises . . . 33
1.8 Pearls of Wisdom . . . 36
1.8.1 Miscellaneous . . . 36
1.8.2 The Concept "Generic" . . . 37
1.8.3 A Comparison Theorem . . . 38

2 Linear Systems with Constant Coefficients **41**

2.1 Preview . . . 41
2.2 Definition and Properties of the Matrix Exponential . . . 42
2.2.1 Preliminaries About Norms . . . 42
2.2.2 Convergence . . . 46
2.2.3 The Main Theorem and Its Proof . . . 48
2.3 Calculation of the Matrix Exponential . . . 50
2.3.1 The Role of Similarity . . . 50
2.3.2 Two Problematic Cases . . . 52
2.3.3 Use of the Jordan Form . . . 55

2.4 Large-Time Behavior of Solutions of Linear Systems . . . 57
2.4.1 The Main Results . . . 57
2.4.2 Tests for Eigenvalues in the Left Half-Plane . . . 59
2.5 Classification and Phase Portraits for 2×2 Systems . . . 60
2.5.1 Saddles and Nodes . . . 60
2.5.2 Foci and Centers . . . 62
2.5.3 Additional Remarks . . . 62
2.6 Solution of Inhomogeneous Problems . . . 63
2.7 Exercises . . . 64
2.7.1 Core Exercises . . . 64
2.7.2 Practice with Linear Algebra . . . 68
2.7.3 Practice Sketching Phase Portraits . . . 70
2.7.4 PHD Exercises . . . 72
2.8 Pearls of Wisdom . . . 74
2.8.1 Alternative Norms . . . 74
2.8.2 Nondifferentiable Limits and the Cantor Set . . . 75
2.8.3 More on Generic Behavior . . . 77

3 Nonlinear Systems: Local Theory **79**

3.1 Two Counterexamples . . . 79
3.2 Local Existence Theory . . . 81
3.2.1 Statement of the Existence Theorem . . . 81
3.2.2 $\mathcal{C}^1$ Implies Lipschitz Continuity . . . 82
3.2.3 Reformulation of the IVP as an Integral Equation . . . 85

3.2.4 The Contraction-Mapping Principle . . . 85
3.2.5 Proof of the Existence Theorem . . . 87
3.2.6 An Illustrative Example and Picard Iteration . . . 90
3.2.7 Concluding Remarks . . . 91
3.3 Uniqueness Theory . . . 91
3.3.1 Gronwall's Lemma . . . 91
3.3.2 More on Lipschitz Functions . . . 93
3.3.3 The Uniqueness Theorem . . . 94
3.4 Generalization to Nonautonomous Systems . . . 96
3.4.1 Nonlinear Systems . . . 96
3.4.2 Linear Systems . . . 96
3.5 Exercises . . . 97
3.5.1 Core Exercises . . . 97
3.5.2 Linear ODEs with Periodic Coefficients . . . 102
3.5.3 PHD Exercises . . . 104
3.6 Pearls of Wisdom . . . 106
3.6.1 Miscellaneous . . . 106
3.6.2 Resonance . . . 108

4 Nonlinear Systems: Global Theory 111

4.1 The Maximal Interval of Existence . . . 112
4.2 Two Sufficient Conditions for Global Existence . . . 113
4.2.1 Linear Growth of the RHS . . . 113
4.2.2 Trapping Regions . . . 114

4.3 Level Sets and Trapping Regions . . . 119
4.3.1 Introduction via Duffing's Equation . . . 119
4.3.2 The Chemostat . . . 119
4.3.3 The Torqued Pendulum and ODEs on Manifolds . . . 120
4.4 Nullclines and Trapping Regions . . . 123
4.4.1 Nullclines in the Chemostat . . . 123
4.4.2 An Activator–Inhibitor System . . . 124
4.4.3 Sel'kov's Model for Glycolysis . . . 127
4.4.4 Van der Pol's Equation . . . 128
4.4.5 Michaelis–Menten Kinetics . . . 129
4.5 Continuity Properties of the Solution . . . 131
4.5.1 The Main Issue: Continuous Dependence on Initial Conditions . . . 131
4.5.2 Some Associated Formalism . . . 133
4.5.3 Continuity with Respect to Parameters . . . 134
4.6 Differentiability Properties of the Solution . . . 134
4.6.1 Dependence on Initial Conditions . . . 134
4.6.2 The Perspective of Differentiability . . . 136
4.6.3 Examples . . . 137
4.6.4 The Order Notation . . . 138
4.6.5 Proof of Theorem 4.6.1 . . . 140
4.6.6 Tying Up Loose Ends . . . 141
4.6.7 Generalizations . . . 141

4.7 Exercises . . . 142
4.7.1 Core Exercises . . . 143
4.7.2 Applying the Differentiation Theorems . . . 149
4.7.3 Some Mopping-Up Exercises . . . 151
4.7.4 Computing Exercise . . . 152
4.7.5 PHD Exercises . . . 153
4.8 Pearls of Wisdom . . . 155
4.9 Appendix: Euler's Method . . . 156
4.9.1 Introduction . . . 156
4.9.2 Theoretical Basis for the Approximation . . . 157
4.9.3 Convergence of the Numerical Solution . . . 158

5 Nondimensionalization and Scaling 161

5.1 Classes of ODEs in Applications . . . 162
5.1.1 Mechanical Models . . . 162
5.1.2 Electrical Models . . . 163
5.1.3 "Bathtub" Models . . . 163
5.2 Scaling Example 0: Two Models from Ecology . . . 165
5.3 Scaling Example 1: A Nonlinear Oscillator . . . 169
5.4 Scaling Example 2: Sel'kov's Model for Glycolysis . . . 172
5.5 Scaling Example 3: The Chemostat . . . 174
5.5.1 ODEs Modeling Flow Through a Reactor . . . 174
5.5.2 The Chemostat . . . 176
5.6 Scaling Example 4: An Activator–Inhibitor System . . . 178

5.7 Scaling Example 5: Michaelis–Menten Kinetics . . . 181
5.8 Exercises . . . 184
5.8.1 Core Exercises . . . 185
5.8.2 Anticipatory Exercises . . . 187
5.8.3 PHD Exercises . . . 189
5.9 Pearls of Wisdom . . . 191
5.9.1 Making Scaling Work for You . . . 191
5.9.2 A Nod to Scientific Literacy . . . 194

6 Trajectories Near Equilibria 195

6.1 Stability of Equilibria . . . 196
6.1.1 The Main Theorem . . . 196
6.1.2 An Easy Application . . . 198
6.2 Terminology to Classify Equilibria . . . 199
6.2.1 Terms Related to Theorem 6.1.1 . . . 199
6.2.2 Other Terms Based on Eigenvalues . . . 201
6.2.3 Section 1.6 Revisited, Part I . . . 202
6.2.4 Two-Dimensional Equilibria and Slopes of Nullclines . . . 205
6.3 Activator–Inhibitor Systems and the Turing Instability . . . 206
6.3.1 Equilibria of the Activator–Inhibitor System . . . 206
6.3.2 The Turing Instability: Destabilization by Diffusion . . . 208
6.4 Feedback Stabilization of an Inverted Pendulum . . . 210
6.5 Lyapunov Functions . . . 215
6.5.1 The Main Results . . . 215

6.5.2 Lasalle's Invariance Principle . . . 217
6.5.3 Construction of Lyapunov Functions: An Example . . . 218
6.6 Stable and Unstable Manifolds . . . 220
6.6.1 A Linear Example . . . 220
6.6.2 Statement of the Local Theorem . . . 222
6.6.3 A Nonlinear Example . . . 223
6.6.4 Global Behavior of Stable/Unstable Manifolds . . . 226
6.7 Drawing Phase Portraits . . . 228
6.7.1 Example 1: The Chemostat . . . 229
6.7.2 Example 2: The Activator–Inhibitor . . . 230
6.7.3 Example 3: Section 1.6 Revisited, Part II . . . 231
6.8 Exercises . . . 233
6.8.1 Core Exercises . . . 233
6.8.2 Uses of a Lyapunov Function . . . 238
6.8.3 Phase-Portrait Exercises . . . 240
6.8.4 PHD Exercises . . . 241
6.9 Pearls of Wisdom . . . 245
6.9.1 Miscellaneous . . . 245
6.9.2 The Hartman–Grobman Theorem and Topological Conjugacy . . . 246
6.9.3 Structural Stability . . . 247
6.10 Appendix 1: Partial Proof of Theorem 6.6.1 . . . 248
6.10.1 Reformulation of the IVP as an Integral Equation . . . 248
6.10.2 Fixed-Point Analysis . . . 250

6.10.3 The Stable Manifold . . . 252
6.10.4 Stable Manifolds at Nonhyperbolic Equilibria . . . 253
6.11 Appendix 2: Center Manifolds and Nonhyperbolicity . . . 254
6.11.1 First Example . . . 255
6.11.2 Second Example . . . 256

7 Oscillations in ODEs 259

7.1 Periodic Solutions . . . 259
7.1.1 Basic Issues . . . 259
7.1.2 Examples of Periodic Solutions . . . 261
7.1.3 A Leisurely Overview of This Chapter . . . 265
7.2 Special Behavior in Two Dimensions . . . Mostly . . . 267
7.2.1 The Poincaré–Bendixson Theorem: Minimal Version . . . 267
7.2.2 Applications of the Theorem . . . 268
7.2.3 Limit Sets (in Any Dimension) . . . 270
7.2.4 The Poincaré–Bendixson Theorem: Strong Version . . . 274
7.2.5 Nonexistence: Dulac's Theorem . . . 275
7.2.6 Section 1.6 Revisited, Part III . . . 275
7.3 Stability of Periodic Orbits and the Poincaré Map . . . 276
7.3.1 An Eigenvalue Test for Stability . . . 276
7.3.2 Basics of the Poincaré Map . . . 278
7.3.3 Discrete-Time Dynamics and the Proof of Theorem 7.3.2 . . . 281
7.4 Stability of the Limit Cycle in the Torqued Pendulum . . . 283

7.5 Van der Pol with Small β: Weakly Nonlinear Analysis 285
7.5.1 Two Illustrative Examples of Perturbation Theory 286
7.5.2 Application to the van der Pol Equation 291
7.6 Van der Pol with Large β: Singular Perturbation Theory 293
7.6.1 Two Sources of Guidance 293
7.6.2 Approximation of the Initial Decay in (7.58) 295
7.6.3 Phase-Plane Analysis of a Related Equation 298
7.6.4 Concluding Remarks 301
7.7 Stability of the van der Pol Limit Cycles 301
7.7.1 Case 1: Small β 302
7.7.2 Case 2: Large β 303
7.8 Exercises 304
7.8.1 Core Exercises 304
7.8.2 The Poincaré–Lindstedt Method 311
7.8.3 Changes of Variables 313
7.8.4 PHD Exercises 316
7.9 Pearls of Wisdom 318
7.9.1 Area and Dulac's Theorem 318
7.9.2 Poincaré-Like Maps in Constructing Periodic Solutions 319
7.9.3 Stable/Unstable Manifolds in Other Contexts 320
7.9.4 Miscellaneous 321
7.10 Appendix: Stabilizing an Inverted Pendulum 321
7.10.1 A Smidgen of Floquet Theory 321

7.10.2 Some Stable Solutions of Mathieu's Equation 323
7.10.3 Application to the Inverted Vibrated Pendulum 325

8 Bifurcation from Equilibria **327**

8.1 Examples of Pitchfork Bifurcation 328
8.1.1 Bead on a Rotating Hoop 328
8.1.2 The Lorenz Equations 330
8.1.3 A Laterally Supported Inverted Pendulum 332
8.2 Perspectives on This Chapter 334
8.2.1 An Outline of the Chapter 334
8.2.2 A Bifurcation Theorem 334
8.3 Examples of Transcritical Bifurcation 336
8.3.1 Section 1.6 Revisited: Part IV 336
8.3.2 The Chemostat 337
8.4 Examples of Saddle-Node Bifurcation 339
8.4.1 The Torqued Pendulum 339
8.4.2 Activator–Inhibitor Systems 340
8.5 The Lyapunov–Schmidt Reduction 341
8.5.1 Bare Bones of the Reduction 341
8.5.2 Proof of Theorem 8.2.2 344
8.5.3 One-Dimensional Bifurcation Problems 348
8.5.4 Exchange of Stability 350
8.5.5 Symmetry and the Pitchfork Bifurcation 351
8.5.6 Additional Parameters in Bifurcation Problems 352

8.6 Steady-State Bifurcation in Two Applications . . . 355
8.6.1 The Two-Cell Turing Instability . . . 355
8.6.2 The CSTR . . . 360
8.7 Examples of Hopf Bifurcation . . . 364
8.7.1 An Academic Example . . . 364
8.7.2 The "Repressilator" . . . 366
8.7.3 Section 1.6 Revisited: Part V . . . 368
8.7.4 The "Denatured" Morris–Lecar System . . . 369
8.8 Theoretical Description of Hopf Bifurcation . . . 372
8.8.1 A Bifurcation Theorem . . . 372
8.8.2 The Activator–Inhibitor: Extreme Nongeneric Behavior . . . 374
8.8.3 Sub/Supercriticality in Two Dimensions . . . 375
8.9 Exercises . . . 376
8.9.1 Core Exercises . . . 376
8.9.2 Applications of Bifurcation Theory . . . 380
8.9.3 PHD Exercises . . . 385
8.10 Pearls of Wisdom . . . 395
8.10.1 Comments on Proving the Hopf Bifurcation Theorem . . . 395
8.10.2 High-Dimensional Bifurcation: Symmetry and Mode Competition . . . 397
8.10.3 Homeostasis, or "Antibifurcation" . . . 399

9 Examples of Global Bifurcation **403**

9.1 Homoclinic Bifurcation . . . 403
9.1.1 An Academic Example . . . 403

9.1.2 The van der Pol Equation with a Nonlinear Restoring Force . . . 405
9.1.3 Section 1.6 Revisited: Part VI . . . 405
9.1.4 Other Examples of Homoclinic and Heteroclinic Bifurcations . . . 408
9.2 Saddle-Node Bifurcation of Limit Cycles . . . 409
9.2.1 An Academic Example . . . 409
9.2.2 The Denatured Morris–Lecar Equation . . . 409
9.2.3 The Overdamped Torqued Pendulum . . . 411
9.3 Poincaré Maps and Stability Loss of Limit Cycles . . . 414
9.4 Mutual Annihilation of Two Limit Cycles . . . 415
9.4.1 An Academic Example . . . 415
9.4.2 The Denatured Morris–Lecar Equation . . . 415
9.4.3 Phase-Locking in Coupled Oscillators . . . 417
9.5 Hopf-Like Bifurcation to an Invariant Torus . . . 417
9.5.1 An Academic Example . . . 417
9.5.2 The Periodically Forced van der Pol Equation . . . 419
9.5.3 Other Examples of Bifurcation to an Invariant Torus . . . 420
9.6 Period-Doubling . . . 420
9.6.1 Academic Example 1: Mappings . . . 422
9.6.2 Cardiac Alternans . . . 423
9.6.3 Academic Example 2: ODEs . . . 426
9.6.4 A Periodically Forced Pendulum . . . 428
9.6.5 Rössler's Equation . . . 430
9.6.6 Other Examples . . . 432

9.7 The Onset of Chaos in the Lorenz Equations . . . 433
9.8 Bursting in the Denatured Morris–Lecar Equations . . . 438
9.9 Exercises . . . 440
9.9.1 Core Exercises . . . 440
9.9.2 Computations to Support Claims in the Text . . . 444
9.9.3 Bifurcation in a Quadratic Map . . . 446
9.9.4 PHD Exercises . . . 447
9.10 Pearls of Wisdom . . . 448
9.10.1 Remarks on Heteroclinic Orbits . . . 448
9.10.2 Bifurcation in Fluid-Mechanics Problems . . . 449
9.10.3 Routes to Chaos . . . 449

10 Epilogue **451**

10.1 Boundary Value Problems . . . 451
10.1.1 An Overview Through Examples . . . 451
10.1.2 Eigenvalue Problems . . . 455
10.2 Stochastic Population Models . . . 457
10.3 Numerical Methods: Two Sobering Examples . . . 460
10.3.1 Stiff ODEs . . . 460
10.3.2 Unreasonable Behavior of Reasonable Methods . . . 461
10.4 ODEs on a Torus: Entrainment . . . 463
10.5 Delay Differential Equations . . . 467

10.6 A Peek at Chaos . . . 470
10.6.1 A One-Dimensional Mapping Model . . . 470
10.6.2 The Lorenz Equations . . . 473
10.7 Exercises . . . 479
10.7.1 Core Exercises . . . 479
10.7.2 PHD Exercises . . . 482
10.8 Pearls of Wisdom . . . 484
10.8.1 The Elastica . . . 484
10.8.2 A Bit More on Chaos . . . 486

A Guide to Commonly Used Notation **487**

A.1 Letter Choices . . . 487
A.2 Other Notations . . . 490
A.3 Other Conventions . . . 491

B Notions from Advanced Calculus **493**

B.1 Basic Issues . . . 493
B.2 Pointwise and Uniform Convergence . . . 495
B.2.1 Sequences . . . 495
B.2.2 Series . . . 497
B.2.3 Convergence of Integrals . . . 499
B.3 Selected Issues in Vector Calculus . . . 499
B.3.1 Differentiability . . . 500
B.3.2 The Implicit Function Theorem . . . 501
B.3.3 Surfaces and Manifolds . . . 503

B.4 Exercises 50
B.4.1 Core Exercises 50
B.4.2 PHD Exercises 51
B.5 Pearls of Wisdom 51

C Notions from Linear Algebra **51**

C.1 How to Work with Jordan Normal Forms 51
C.2 The Real Canonical Form of a Matrix 51
C.3 Eigenvalues as Continuous Functions of Matrix Entries 52
C.4 The Routh–Hurwitz Criterion 52
C.5 Exercises 52
C.5.1 Core Exercises 52
C.5.2 PHD Exercises 52

Bibliography **52**

Index **53**