Contents

Preface to the Second Edition vii

Preface to the First Edition ix

GENERAL INTRODUCTION.
The User's Guide 1

Introduction 1
1. Mechanism and Description of Chaos. The Finite-Dimensional Case 2
2. Mechanism and Description of Chaos. The Infinite-Dimensional Case 6
3. The Global Attractor. Reduction to Finite Dimension 10
4. Remarks on the Computational Aspect 12
5. The User's Guide 13

CHAPTER I
General Results and Concepts on Invariant Sets and Attractors 15

Introduction 15
1. Semigroups, Invariant Sets, and Attractors 16
1.1. Semigroups of Operators 16
1.2. Functional Invariant Sets 18
1.3. Absorbing Sets and Attractors 20
1.4. A Remark on the Stability of the Attractors 28
2. Examples in Ordinary Differential Equations 29
2.1. The Pendulum 29
2.2. The Minea System 32
2.3. The Lorenz Model 34
3. Fractal Interpolation and Attractors 36
3.1. The General Framework 37
3.2. The Interpolation Process 38
3.3. Proof of Theorem 3.1 40

CHAPTER II
Elements of Functional Analysis 43

Introduction 43
1. Function Spaces 43
1.1. Definition of the Spaces. Notations 43
1.2. Properties of Sobolev Spaces 45
1.3. Other Sobolev Spaces 49
1.4. Further Properties of Sobolev Spaces 51
2. Linear Operators 53
2.1. Bilinear Forms and Linear Operators 54
2.2. "Concrete" Examples of Linear Operators 58
3. Linear Evolution Equations of the First Order in Time 68
3.1. Hypotheses 68
3.2. A Result of Existence and Uniqueness 70
3.3. Regularity Results 71
3.4. Time-Dependent Operators 74
4. Linear Evolution Equations of the Second Order in Time 76
4.1. The Evolution Problem 76
4.2. Another Result 79
4.3. Time-Dependent Operators 80

CHAPTER III
Attractors of the Dissipative Evolution Equation of the First Order in Time: Reaction–Diffusion Equations. Fluid Mechanics and Pattern Formation Equations 82

Introduction 82
1. Reaction–Diffusion Equations 83
1.1. Equations with a Polynomial Nonlinearity 84
1.2. Equations with an Invariant Region 93
2. Navier–Stokes Equations ($n = 2$) 104
2.1. The Equations and Their Mathematical Setting 105
2.2. Absorbing Sets and Attractors 109
2.3. Proof of Theorem 2.1 113
3. Other Equations in Fluid Mechanics 115
3.1. Abstract Equation. General Results 115
3.2. Fluid Driven by Its Boundary 118
3.3. Magnetohydrodynamics (MHD) 123
3.4. Geophysical Flows (Flows on a Manifold) 127
3.5. Thermohydraulics 133
4. Some Pattern Formation Equations 141
4.1. The Kuramoto–Sivashinsky Equation 141
4.2. The Cahn–Hilliard Equation 151
5. Semilinear Equations 162
5.1. The Equations. The Semigroup 162
5.2. Absorbing Sets and Attractors 167
5.3. Proof of Theorem 5.2 170

6. Backward Uniqueness 171
6.1. An Abstract Result 172
6.2. Applications 175

CHAPTER IV
Attractors of Dissipative Wave Equations 179

Introduction 179
1. Linear Equations: Summary and Additional Results 180
1.1. The General Framework 181
1.2. Exponential Decay 183
1.3. Bounded Solutions on the Real Line 186
2. The Sine–Gordon Equation 188
2.1. The Equation and Its Mathematical Setting 189
2.2. Absorbing Sets and Attractors 191
2.3. Other Boundary Conditions 196
3. A Nonlinear Wave Equation of Relativistic Quantum Mechanics 202
3.1. The Equation and Its Mathematical Setting 202
3.2. Absorbing Sets and Attractors 206
4. An Abstract Wave Equation 212
4.1. The Abstract Equation. The Group of Operators 212
4.2. Absorbing Sets and Attractors 215
4.3. Examples 220
4.4. Proof of Theorem 4.1 (Sketch) 224
5. The Ginzburg–Landau Equation 226
5.1. The Equations and Its Mathematical Setting 227
5.2. Absorbing Sets and Attractors 230
6. Weakly Dissipative Equations. I. The Nonlinear Schrödinger Equation 234
6.1. The Nonlinear Schrödinger Equation 235
6.2. Existence and Uniqueness of Solution. Absorbing Sets 236
6.3. Decomposition of the Semigroup 239
6.4. Comparison of z and Z for Large Times 250
6.5. Application to the Attractor. The Main Result 252
6.6. Determining Modes 254
7. Weakly Dissipative Equations II. The Korteweg–De Vries Equation 256
7.1. The Equation and its Mathematical Setting 257
7.2. Absorbing Sets and Attractors 260
7.3. Regularity of the Attractor 269
7.4. Proof of the Results in Section 7.1 272
7.5. Proof of Proposition 7.2 290
8. Unbounded Case: The Lack of Compactness 306
8.1. Preliminaries 307
8.2. The Global Attractor 312
9. Regularity of Attractors 316
9.1. A Preliminary Result 317
9.2. Example of Partial Regularity 322
9.3. Example of $\mathscr{C}^\infty$ Regularity 324
10. Stability of Attractors 329

CHAPTER V
Lyapunov Exponents and Dimension of Attractors 335

Introduction 335
1. Linear and Multilinear Algebra 336
1.1. Exterior Product of Hilbert Spaces 336
1.2. Multilinear Operators and Exterior Products 340
1.3. Image of a Ball by a Linear Operator 347
2. Lyapunov Exponents and Lyapunov Numbers 355
2.1. Distortion of Volumes Produced by the Semigroup 355
2.2. Definition of the Lyapunov Exponents and Lyapunov Numbers 357
2.3. Evolution of the Volume Element and Its Exponential Decay: The Abstract Framework 362
3. Hausdorff and Fractal Dimensions of Attractors 365
3.1. Hausdorff and Fractal Dimensions 365
3.2. Covering Lemmas 367
3.3. The Main Results 368
3.4. Application to Evolution Equations 377

CHAPTER VI
Explicit Bounds on the Number of Degrees of Freedom and the Dimension of Attractors of Some Physical Systems 380

Introduction 380
1. The Lorenz Attractor 381
2. Reaction–Diffusion Equations 385
2.1. Equations with a Polynomial Nonlinearity 386
2.2. Equations with an Invariant Region 392
3. Navier–Stokes Equations ($n = 2$) 397
3.1. General Boundary Conditions 398
3.2. Improvements for the Space-Periodic Case 404
4. Other Equations in Fluid Mechanics 412
4.1. The Linearized Equations (The Abstract Framework) 412
4.2. Fluid Driven by Its Boundary 413
4.3. Magnetohydrodynamics 420
4.4. Flows on a Manifold 425
4.5. Thermohydraulics 430
5. Pattern Formation Equations 434
5.1. The Kuramoto–Sivashinsky Equation 435
5.2. The Cahn–Hilliard Equations 441
6. Dissipative Wave Equations 446
6.1. The Linearized Equation 447
6.2. Dimension of the Attractor 450
6.3. Sine–Gordon Equations 453
6.4. Some Lemmas 454
7. The Ginzburg–Landau Equation 456
7.1. The Linearized Equation 456
7.2. Dimension of the Attractor 457
8. Differentiability of the Semigroup 461

CHAPTER VII
Non-Well-Posed Problems, Unstable Manifolds, Lyapunov Functions, and Lower Bounds on Dimensions 465

Introduction 465

PART A: NON-WELL-POSED PROBLEMS 466

1. Dissipativity and Well Posedness 466
1.1. General Definitions 466
1.2. The Class of Problems Studied 467
1.3. The Main Result 471
2. Estimate of Dimension for Non-Well-Posed Problems: Examples in Fluid Dynamics 475
2.1. The Equations and Their Linearization 476
2.2. Estimate of the Dimension of X 477
2.3. The Three-Dimensional Navier–Stokes Equations 479

PART B: UNSTABLE MANIFOLDS, LYAPUNOV FUNCTIONS, AND LOWER BOUNDS ON DIMENSIONS 482
3. Stable and Unstable Manifolds 482
3.1. Structure of a Mapping in the Neighborhood of a Fixed Point 483
3.2. Application to Attractors 485
3.3. Unstable Manifold of a Compact Invariant Set 489
4. The Attractor of a Semigroup with a Lyapunov Function 490
4.1. A General Result 490
4.2. Additional Results 492
4.3. Examples 495
5. Lower Bounds on Dimensions of Attractors: An Example 496

CHAPTER VIII
The Cone and Squeezing Properties. Inertial Manifolds 498

Introduction 498
1. The Cone Property 499
1.1. The Cone Property 499
1.2. Generalizations 502
1.3. The Squeezing Property 504
2. Construction of an Inertial Manifold: Description of the Method 505
2.1. Inertial Manifolds: The Method of Construction 505
2.2. The Initial and Prepared Equations 506
2.3. The Mapping $\mathscr{T}$ 509
3. Existence of an Inertial Manifold 512
3.1. The Result of Existence 513
3.2. First Properties of $\mathscr{T}$ 514
3.3. Utilization of the Cone Property 516
3.4. Proof of Theorem 3.1 (End) 522
3.5. Another Form of Theorem 3.1 525
4. Examples 526
4.1. Example 1: The Kuramoto–Sivashinsky Equation 526

4.2. Example 2: Approximate Inertial Manifolds for the Navier–Stokes Equations 528
4.3. Example 3: Reaction–Diffusion Equations 530
4.4. Example 4: The Ginzburg–Landau Equation 531
5. Approximation and Stability of the Inertial Manifold with Respect to Perturbations 532

CHAPTER IX
Inertial Manifolds and Slow Manifolds. The Non-Self-Adjoint Case 536

Introduction 536
1. The Functional Setting 537
1.1. Notations and Hypotheses 537
1.2. Construction of the Inertial Manifold 539
2. The Main Result (Lipschitz Case) 541
2.1. Existence of Inertial Manifolds 541
2.2. Properties of $\mathscr{T}$ 542
2.3. Smoothness Property of Φ (Φ is $\mathscr{C}^1$) 548
2.4. Proof of Theorem 2.1 550
3. Complements and Applications 553
3.1. The Locally Lipschitz Case 553
3.2. Dimension of the Inertial Manifold 555
4. Inertial Manifolds and Slow Manifolds 559
4.1. The Motivation 559
4.2. The Abstract Equation 560
4.3. An Equation of Navier–Stokes Type 562

CHAPTER X
Approximation of Attractors and Inertial Manifolds. Convergent Families of Approximate Inertial Manifolds 565

Introduction 565
1. Construction of the Manifolds 566
1.1. Approximation of the Differential Equation 566
1.2. The Approximate Manifolds 569
2. Approximation of Attractors 571
2.1. Properties of $\mathscr{T}_N^\tau$ 571
2.2. Distance to the Attractor 573
2.3. The Main Result 576
3. Convergent Families of Approximate Inertial Manifolds 578
3.1. Properties of $\mathscr{T}_N^\tau$ 579
3.2. Distance to the Exact Inertial Manifold 581
3.3. Convergence to the Exact Inertial Manifold 583

APPENDIX
Collective Sobolev Inequalities 585

Introduction 585
1. Notations and Hypotheses 586
1.1. The Operator $\mathfrak{U}$ 586
1.2. The Schrödinger-Type Operators 588

2. Spectral Estimates for Schrödinger-Type Operators 590
2.1. The Birman–Schwinger Inequality 590
2.2. The Spectral Estimate 593
3. Generalization of the Sobolev–Lieb–Thirring Inequality (I) 596
4. Generalization of the Sobolev–Lieb–Thirring Inequality (II) 602
4.1. The Space-Periodic Case 603
4.2. The General Case 605
4.3. Proof of Theorem 4.1 607
5. Examples 610

Bibliography 613

Index 645